

Gauge invariant gluon spin operator for spinless non-linear wave solutions

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We consider non-linear wave type solutions with intrinsic mass scale parameter and zero spin in a pure $SU(2)$ quantum chromodynamics (QCD). A new stationary solution which can be treated as a system of a static Wu-Yang monopole dressed in off-diagonal gluon field is proposed. A remarkable feature of such a solution is that it possesses a finite energy density everywhere. All considered non-linear wave type solutions have common features: presence of the mass scale parameter, non-vanishing projection of the color fields along the propagation direction and zero spin. The last property requires revision of the gauge invariant definition of the spin density operator which supposed to produce spin one states for the massless vector gluon field. We construct a gauge invariant definition of the classical gluon spin density operator which is unique and Lorentz frame independent.

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I. INTRODUCTION

There is a common belief that the color confinement and the related mass gap problem in quantum chromodynamics (QCD) need a consistent non-perturbative quantum theory for their resolution [1]. A so-called proton spin crisis [2–4] represents another puzzle which is closely related to non-perturbative dynamics of constituent quarks and gluons. Besides, recent studies reveal deep conceptual problems in definitions of the momentum, spin and orbital angular momentum operators of quarks and gluons (see [5] for a review and references there in). In this respect, an important step towards a strict and non-perturbative theory of QCD is to describe the dynamical content of classical non-perturbative solutions in a pure $SU(2)$ Yang-Mills theory and establish their relationship to fundamental observable quantities in QCD such as vacuum gluon condensate, glueball spectrum and others.

Non-linear structure and rich topology of the Yang-Mills theory lead to a wide class of various exact solutions [6]. Monopoles and instantons represent the most well-known topological solutions which have numerous physical implications (see, for ex., [7, 8]). Much less is known about non-linear wave type solutions and especially about their physical meaning. Non-linear transverse plane wave solutions representing analogues of the

electro-magnetic plane waves were found in 80s by Coleman [9]. The existence of another type of non-linear plane wave solutions with a mass scale parameter [10] is an important manifestation of the conformal symmetry in the Yang-Mills theory. Various representations of the non-linear plane waves and some special non-linear spherical wave solutions were considered in [11–15]. There is a hope that the knowledge of full structure of non-linear wave solutions in the Yang-Mills theory can provide a novel approach towards non-perturbative description of quantum chromodynamics.

In the present paper we undertake an attempt to describe a special class of stationary non-linear wave solutions in a pure $SU(2)$ Yang-Mills theory (or QCD) which admit intrinsic mass scale parameter. Among such solutions there are non-linear stationary plane wave solutions [10] and non-stationary spherically symmetric solutions [16–20] which resemble kink type solitons in two-dimensional coordinate plane (t, r) . It is known that in a pure Yang-Mills theory a stable solitonic solutions do not exist due to the Coleman theorem [21]. This implies that any wave packet solution with a finite total energy must decay by radiating its energy to space infinity. It is surprising that a regular stationary monopole like solution with a finite energy density everywhere does exist even in a pure $SU(2)$ QCD without introducing any additional matter fields. We demonstrate the existence of a class of such regular stationary solutions which represent a system of the Wu-Yang monopole interacting to the off-diagonal gluons in a special gauge. All considered non-linear propagating solutions and the new proposed stationary monopole like solutions possess intriguing properties, namely, they admit a mass scale parameter, non-vanishing longitudinal projections of the color

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fields and vanished classical spin density. This indicates to existence of massive spinless states in the quantum theory. Such an idea, that stationary solutions may describe particles (or pseudo-particles) and even might be related to the vacuum structure in QCD, was sounded long time ago [22–24]. Certainly, to obtain rigorous description of particle spectrum based on standard QCD as a fundamental theory of strong interaction one has to construct a consistent non-perturbative quantization scheme for QCD which remains an unresolved problem so far. The existence of the non-linear wave type solutions with a non-zero mass and vanished spin manifests inconsistency of the perturbative QCD which has massless vector gluons in the initial Fock space of physical states. So even at classical level one should formulate a strict notion of the gluon spin density operator on the class of massive non-linear wave solutions. A consistent definition of the gluon spin and angular momentum operators in QCD represents unresolved problem and there is still controversy between different approaches to this problem [5]. In particular, there is no unique gauge invariant and at the same time explicitly Lorentz frame independent definition for the gluon spin and orbital momentum operators. In the present paper we show that for the stationary non-linear wave solutions with the mass parameter it is possible to construct a unique gauge-invariant and Lorentz frame independent gluon spin density operator at classical level.

II. NON-LINEAR PROPAGATING SOLUTIONS WITH A MASS SCALE PARAMETER

In this section we overview briefly the known non-linear propagating wave type solutions which possess a mass scale parameter and non-vanishing longitudinal projections of color fields [10, 16–20].

1. Non-linear spinless plane waves

We consider a class of non-linear wave solutions admitting mass scale parameters in the case of a pure $SU(2)$ Yang-Mills theory. The respective Lagrangian and equations of motion are as follows

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{a\mu\nu}F^{a\mu\nu}, \\ (D^\mu F_{\mu\nu})^a &\equiv \partial^\mu F_{\mu\nu}^a + g\epsilon^{abc}A^{b\mu}F_{\mu\nu}^c = 0, \end{aligned} \quad (1)$$

where the field strength is given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c$$

with the group structure constants ϵ^{abc} and the coupling constant g . The known Coleman non-linear transverse plane wave solutions [9] form a family with six arbitrary functions depending on light-cone coordinates which correspond to six transverse propagating massless gluon

modes in agreement with perturbative description of gluons in QCD. An important feature of the non-Abelian gauge theory is that it admits a three-parametric family of non-linear massive wave solutions with a field strength having non-vanishing longitudinal projections along the wave vector [11–14]. Such plane waves can be interpreted as non-perturbative longitudinal modes of gluon. The solutions can be reproduced by using a simple ansatz

$$A_i^a(x) = \delta_i^a \phi_i(u), \quad A_0^3(x) = \beta \phi_3(u), \quad (2)$$

where $u \equiv k_0 t + k_3 z$ and $\beta = v/c$ is a kinematic variable proportional to wave velocity “ v ” in units of the speed of light “ c ”. The non-vanishing field strength components are the following:

$$\begin{aligned} F_{10}^1 &= -\partial_0 \phi_1, & F_{20}^1 &= g\beta \phi_2 \phi_3, \\ F_{10}^2 &= -g\beta \phi_1 \phi_3, & F_{20}^2 &= -\partial_0 \phi_2, \\ F_{30}^3 &= -(1 - \beta^2)\partial_0 \phi_3, \\ F_{13}^1 &= -\beta \partial_0 \phi_1, & F_{13}^2 &= -g\phi_1 \phi_3, \\ F_{23}^1 &= g\phi_2 \phi_3, & F_{23}^2 &= -\beta \partial_0 \phi_2, \\ F_{12}^3 &= g\phi_1 \phi_2. \end{aligned} \quad (3)$$

One can verify that ansatz (2) leads to non-vanishing electric and magnetic longitudinal projections. By direct substitution of the ansatz into the Yang-Mills equations and imposing a constraint $k_3 = \beta k_0$ one can reduce all equations of motion to three ordinary differential equations (ODE)

$$\begin{aligned} k^2 \frac{d^2 \phi_1}{du^2} + g^2 \phi_1 (\phi_2^2 + (1 - \beta^2) \phi_3^2) &= 0, \\ k^2 \frac{d^2 \phi_2}{du^2} + g^2 \phi_2 (\phi_1^2 + (1 - \beta^2) \phi_3^2) &= 0, \\ k^2 \frac{d^2 \phi_3}{du^2} + g^2 \phi_3 (\phi_1^2 + \phi_2^2) &= 0, \end{aligned} \quad (4)$$

where $k^2 \equiv k_0^2 - k_3^2$. Note that the constraint $k_3 = \beta k_0$ provides a Lorenz gauge condition $\partial^\mu A_\mu^a = 0$ for the gauge potential. In the rest frame, $\beta = 0$, the equations (4) describe a classical mechanical system of three anharmonic oscillators with the following Hamiltonian

$$H = \frac{1}{2}(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2) + \frac{g^2}{2}(\phi_1^2 \phi_2^2 + \phi_2^2 \phi_3^2 + \phi_3^2 \phi_1^2). \quad (5)$$

In general, such a system represents a non-integrable problem [25]. It has been proved as well that the corresponding quantum mechanical system possesses a pure discrete energy spectrum despite the presence of zero energy valleys in the Hamiltonian [26].

Let us consider two special cases when the system of equations (4) becomes integrable. The first case, (I), corresponds to a constraint $\phi_1 = \phi_2 \equiv \phi$, $\phi_3 = 0$, and, the second case, (II), is determined by setting $\phi_1 = \phi_2 = \phi_3 \equiv \phi$ and imposing an additional condition $\beta = 0$.

Corresponding equations are the following

$$\begin{aligned} \text{(I)} : \quad & k^2 \frac{d^2 \phi(u)}{du^2} + g^2 \phi^3(u) = 0, \\ \text{(II)} : \quad & k^2 \frac{d^2 \phi(u)}{du^2} + 2g^2 \phi^3(u) = 0. \end{aligned} \quad (6)$$

Various representations of these equations have been obtained in $SU(2)$ Yang-Mills theory by using different methods [11–14, 25]. Solutions to the equation (6) are given by the Jacobi elliptic function

$$\begin{aligned} \phi_{\text{I}}(u) &= \frac{\sqrt{2}\mu_1}{\sqrt{g}} \operatorname{sn} \left[\frac{\mu_1}{k} \sqrt{g}(u + u_{01}), -1 \right], \\ \phi_{\text{II}}(u) &= \frac{\mu_2}{\sqrt{g}} \operatorname{sn} \left[\frac{\mu_2}{k} \sqrt{g}(u + u_{02}), -1 \right], \end{aligned} \quad (7)$$

where μ_i, u_{0i} are integration constants. One can set $u_{0i} = 0$ since u_{0i} is the parameter corresponding to translation invariance. The argument of the elliptic function can be re-written in a Lorentz invariant form as $p^\mu x_\mu$ which implies a dispersion relation $p^2 = \mu^2 g$. One should stress that two solutions (7) are gauge non-equivalent and have different implementations in the mathematical structure of the Yang-Mills theory.

2. Spherically symmetric non-stationary finite energy solutions

In this subsection we overview briefly known spherically-symmetric non-stationary solutions [16–20]. We consider a restricted class of such solutions which have a spherically symmetric initial shape in the rest frame, i.e., the total linear momentum is supposed to be zero. A class of finite energy solutions is described by the following ansatz for the gauge potential in spherical coordinates (r, θ, φ) with one unconstrained function $\psi(t, r)$ [16–20]

$$A_m^a = -\epsilon^{abc} \hat{n}^b \partial_m \hat{n}^c (1 + \psi(t, r)), \quad (8)$$

where $\hat{n} = \vec{r}/r$. It is clear, that the ansatz (8) describes a generalized time dependent Wu-Yang monopole configuration. A known static Wu-Yang monopole corresponds to a special case $\psi(t, r) = 0$, and a trivial pure gauge vacuum configuration is described by $\psi(t, r) = \pm 1$. Direct substitution of the ansatz into the equations of motion leads to one non-trivial independent partial differential equation

$$\partial_t^2 \psi - \partial_r^2 \psi + \frac{1}{r^2} \psi (g^2 \psi^2 - 1) = 0. \quad (9)$$

The equation (9) represents a second order hyperbolic equation which admits a wide class of wave solutions determined uniquely by given initial conditions. We give one example of finite energy solutions which looks like

a kink in two-dimensional coordinate plane (t, r) (others can be found in [16–20]). To find a numeric solution one chooses an initial profile function $\psi(t = 0, r)$ in such a manner that the magnetic field vanishes at space infinity $\psi(t = 0, r) = 1 - r^2 e^{-r^2}$. This profile function describes a monopole like configuration with a maximal magnetic charge at a finite distance from the center, and a vanishing total magnetic charge at large distances. A corresponding numeric solution demonstrates a soliton structure in the effective 1+1 space-time (t, r) ; the solution $\psi(t, r)$ represents a lump in the coordinate plane (t, r) , FIG. 5, which moves in radial direction to space infinity with the light speed. Note that one finds a sim-

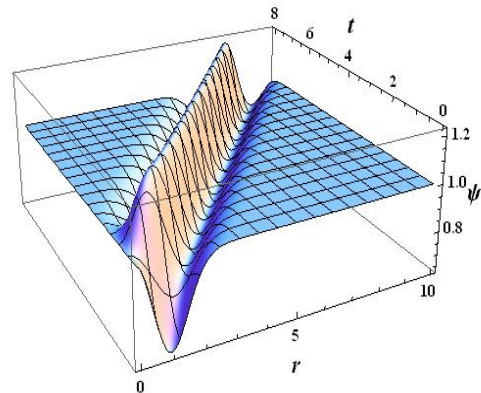


FIG. 1: A kink type solution with initial conditions $\psi(0, r) = 1 - r^2 e^{-r^2}$, $\partial_t \psi(0, r) = 0$.

ilar behavior of the magnetic field for the system of localized monopoles and antimonopoles in the Weinberg-Salam model [27].

We conclude that a pure QCD admits a wide class of non-linear wave solutions. All solutions carry zero spin and the non-linear plane waves has the dispersion relation with the mass parameter. Taking into account that a general solution has additional three parameters corresponding to orientation in the internal space of the group $SU(2)$, one can interpret the non-linear plane waves as three longitudinal dynamic degrees of freedom in addition to six transverse dynamic degrees of freedom represented by Coleman non-Abelian plane waves [9]. Note that transverse non-linear waves remain massless, so the mass appears only from the solutions containing longitudinal projections of the field strength. This is contrary to other approaches like the models of massive gluodynamics where the gluon mass is introduced either through spontaneous symmetry breaking or by adding explicit mass terms [28]. A principal advantage of our treatment is that the color gauge symmetry remains unbroken, so the theory is still renormalizable in a standard perturbative quantum field theory.

III. WU-YANG MONOPOLE DRESSED IN OFF-DIAGONAL GLUON FIELD

Let us rewrite the ansatz (8) by performing an appropriate $SU(2)$ gauge transformation in a so-called Abelian gauge [29] as follows

$$\begin{aligned} A_\theta^2 &= \psi(t, r), \\ A_\varphi^1 &= -\psi(t, r) \sin \theta, \\ A_\varphi^3 &= \frac{1}{g} \cos \theta. \end{aligned} \quad (10)$$

All other components of the gauge potential are identically zero. The expression for the Abelian component A_φ^3 contains coordinate singularity along the z -axis, so such a gauge represents a singular gauge. One should stress, that final gauge-invariant quantities (like the Lagrangian and energy density) are regular and do not depend on a chosen gauge. We prefer the ansatz written in the Abelian gauge [29], (10), since such notation is more suitable for description of stationary monopole solutions in $SU(N)$ Yang-Mills theory and multimonopole configurations. Besides, due to gauge invariant decomposition of the gauge potential [29] one can treat the Abelian gauge potential A_φ^3 as a static Wu-Yang monopole field and the function $\psi(t, r)$ as a dynamic degree of freedom describing the off-diagonal gluon.

We are interested in regular stationary wave solutions to equation (9). Let us write down the energy functional corresponding to the ansatz (10),

$$\begin{aligned} E &= \int dr d\theta d\varphi \sin \theta \left((\partial_t \psi)^2 + (\partial_r \psi)^2 \right. \\ &\quad \left. + \frac{1}{2g^2 r^2} (g^2 \psi^2 - 1)^2 \right) \\ &\equiv 4\pi \int dr \mathcal{E}(t, r), \end{aligned} \quad (11)$$

where $\mathcal{E}(t, r)$ is an effective energy density defined on (1+1)-dimensional half-plane ($r \geq 0$, $0 < t < \infty$). The effective energy density can be treated as a Hamiltonian of two-dimensional $\lambda\phi^4$ theory with a radial dependent coupling $\lambda \equiv 1/(2g^2 r^2)$.

The field strength components contain the following non-vanishing projections of the color magnetic and electric field

$$\begin{aligned} F_{r\theta}^2 &= \partial_r \psi, & F_{r\varphi}^1 &= -\partial_r \psi \sin \theta, \\ F_{\theta\varphi}^3 &= \frac{1}{g} (g^2 \psi^2 - 1) \sin \theta, \\ F_{t\theta}^2 &= \partial_t \psi, & F_{t\varphi}^1 &= -\partial_t \psi \sin \theta. \end{aligned} \quad (12)$$

The radial magnetic field component $F_{\theta\varphi}^3$ generates a non-zero magnetic flux through a sphere with a center at the origin, $r = 0$. So that, the color magnetic charge of the monopole depends on time and distance from the center. Note that various static generalized Wu-Yang

monopoles have been considered before, however, all of them have singularities in agreement with the Derrick's theorem [22].

The equation (9) admits a local non-static solution near the origin which removes the singularity of the monopole at the center

$$\begin{aligned} \psi(t, r) &= \frac{1}{g} + \sum_{n=1} c_{2n}(t) r^{2n}, \\ c_4(t) &= \frac{1}{10} (3g c_2^2(t) + c_2''(t)), \\ c_6(t) &= \frac{1}{28} (c_4''(t) + 6g c_2(t) c_4(t) + g^2 c_2^3(t)), \\ &\vdots \end{aligned} \quad (13)$$

where the coefficient functions $c_{2n}(t)$ ($n \geq 2$) are determined in terms of one arbitrary function $c_2(t)$. In asymptotic region, $r \simeq \infty$, the non-linear equation (9) reduces to a free D'Alembert equation which has a standing spherical wave solution

$$\psi(t, r) \simeq a_0 + A_0 \cos(Mr) \sin(Mt) + \mathcal{O}\left(\frac{1}{r}\right), \quad (14)$$

where a_0, A_0 are integration constants, the mass scale M appears due to scaling invariance in the theory under dilatations $r \rightarrow Mr, t \rightarrow Mt$. One should stress, that the asymptotic solution represents a standing spherical wave only in the leading order and our solution can not be treated as a superposition of out-going and in-going spherical waves due to absence of superposition principle in the non-linear theory. Besides, as we will see below, the parameters a_0, A_0 are not independent free parameters. Moreover, all known before non-linear spherical wave solutions in the Yang-Mills theory are singular. Note that the series expansion for the local solution starts from the factor $1/g^2$ which reflects interrelationship of the solution with the non-perturbative topological origin of the Wu-Yang monopole. In particular, the presence of such a term cancels the singularity of the Wu-Yang monopole. The second term in the series expansion, $c_2(t)r^2$, contains quadratic dependence on the radial coordinate. This provides finiteness of the energy density at the origin.

The equation (9) represents a hyperbolic partial differential equation and admits a correct Cauchy problem setup with arbitrary initial conditions for the function $\psi(t, r)$ at the initial time $t = 0$. For instance, one can choose any initial profile function periodic along the radial direction. However, such a solution will not be stationary in general. To find a stationary solution we will solve a Cauchy problem imposing initial conditions with a periodic in time initial conditions at the origin $r = 0$. To solve numerically such problem one chooses a rectangular numeric domain ($L_0 \leq r \leq L$, $0 \leq t \leq L$). Since at $r = 0$ one has coordinate singularity which implies the stiffness numeric problem we introduce a small number L_0 and then check convergence of the numeric solution

in the limit $L_0 \rightarrow 0$. We use the local solution (13) to impose initial Dirichlet conditions along the boundary $r = L_0$. The initial profile function $c_2(t)$ can be chosen arbitrarily as any regular periodic function, we set it for simplicity in terms of ordinary sine function

$$c_2(t) = c_0 + c_1 \sin(Mt), \quad (15)$$

where c_0, c_1 are numeric constants. Note that only one of two parameters in the local solution (15) is free, the other is fixed by the requirement that a numeric solution matches the asymptotic solution (14). Dimensional analysis implies that the energy of the solutions is proportional to M so that the energy vanishes in the limit $M \rightarrow 0$. This might cause some doubts on existence of the solution. However, one should stress that standard arguments on existence of static solitonic solutions based on the Derrick's theorem [22] are not applicable to the case of stationary solutions which satisfy the extremum principle of the classical action, not the energy functional. In the case of Yang-Mills theory the action is conformal invariant and its first variational derivative with respect to the scale parameter M vanishes identically. So the parameter M determines the scale of the space-time coordinates and can be set to one without loss of generality. With this one can solve numerically the equation (9), and the corresponding solution is depicted in Fig.2. Note that one has a stiffness numeric problem near the

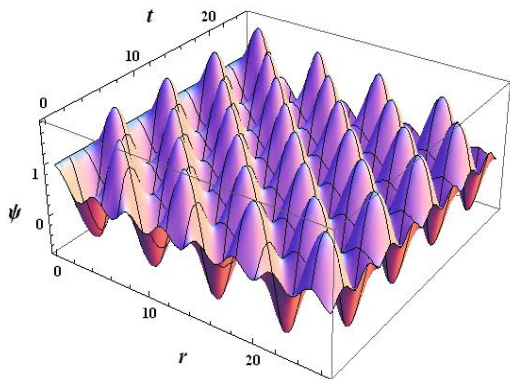


FIG. 2: Stationary monopole solution; $(0 \leq r, t \leq L)$, $L = 8\pi$, $c_0 = -0.041$, $c_1 = -0.523$, $g = 1$.

origin, so that we have checked the regular structure and convergence of the numeric solution in close vicinity of the origin up to $L_0 = 10^{-6}$ while keeping the radial size of the numeric domain of order $L = 64\pi$.

A general stationary monopole solution can be classified by two of three parameters a_0, A_0, M characterizing the asymptotic behavior of the solution (14). The mass scale parameter M takes arbitrary values whereas the mean value a_0 and amplitude A_0 parameters are constrained. Numerical analysis implies the following dependence of the amplitude A_0 of the monopole solution on its mean value parameter a_0 , Fig. 3. The dependence of the amplitude A_0 on the mean value a_0 of the monopole

solution is important in study of the quantum stability of the vacuum gluon condensation based on using the classical stationary monopole solution [30].

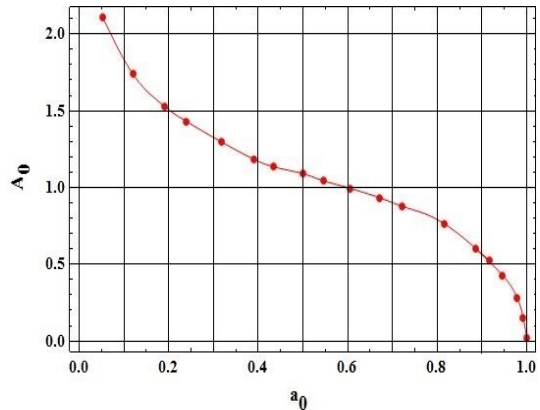


FIG. 3: Dependence of the amplitude A_0 of the monopole solution on the mean value a_0 , $M = 1$.

Note that in the case of vanishing function $\psi(t, r)$ one has still a non-trivial solution for the Wu-Yang monopole which has a singularity at the origin $r = 0$. The monopole is defined by the Abelian gluon field $A_\varphi^3 = \cos \theta$ and the function $\psi(t, r)$ describes off-diagonal gluons. The obtained solution can be treated as a Wu-Yang monopole dressed in a spherical standing wave made of off-diagonal gluons. It is surprising that the spherical standing wave regularizes the singularity of the Wu-Yang monopole resulting in a finite energy density at the origin $r = 0$. Another interesting feature of the solution is that the standing wave does not screen completely the color monopole charge at large distances. One can find that in the asymptotic region the function $\psi(t, r)$ oscillates around the value $b_0 \simeq 0.65$. So the radial component of the color magnetic field $F_{\theta\varphi}^1$ has a non-zero averaged value which provides a non-vanishing total color magnetic charge.

One can easily generalize the above consideration of $SU(2)$ stationary monopole solutions to the case of $SU(N)$ Yang-Mills theory. For the case of a pure $SU(3)$ QCD one has the following ansatz for generalized Wu-Yang monopole solution corresponding to color magnetic charge two

$$\begin{aligned} A_\theta^2 &= \psi_1(t, r), & A_\theta^5 &= \psi_2(t, r), \\ A_\varphi^1 &= -\psi_1(t, r) \sin \theta, & A_\varphi^4 &= \psi_2(t, r) \sin \theta, \\ A_\varphi^3 &= \frac{1}{g} \cos \theta, & A_\varphi^8 &= -\frac{\sqrt{3}}{g} \cos \theta, \end{aligned} \quad (16)$$

where the Abelian gauge potentials $A_\varphi^{(3,8)}$, corresponding to generators of the Cartan subalgebra of $SU(3)$, describe a static Wu-Yang monopole with a total color magnetic charge two, $g_m^{\text{tot}} = 2$, [31]. The functions $\psi_1(t, r)$ and $\psi_2(t, r)$ correspond to dynamic degrees of freedom of the off-diagonal components of the gluon field. One can verify that substitution of the ansatz (16) into the equations

of motion of the pure $SU(3)$ QCD implies two independent partial differential equations for two functions $\psi_{1,2}$

$$\begin{aligned} \partial_t^2 \psi_1 - \partial_r^2 \psi_1 + \frac{g^2}{2r^2} \psi_1 \left(2\psi_1^2 - \psi_2^2 - \frac{2}{g^2} \right) &= 0, \\ \partial_t^2 \psi_2 - \partial_r^2 \psi_2 + \frac{g^2}{2r^2} \psi_2 \left(2\psi_2^2 - \psi_1^2 - \frac{2}{g^2} \right) &= 0. \end{aligned} \quad (17)$$

In a special case, $\psi_1(t, r) = \psi_2(t, r) \equiv \psi(t, r)$, the equations (17) reduce to one differential equation

$$\partial_t^2 \psi - \partial_r^2 \psi + \frac{g^2}{2r^2} \psi \left(\psi^2 - \frac{2}{g^2} \right) = 0. \quad (18)$$

By simple rescaling $\psi(t, r) \rightarrow \sqrt{2}\psi(t, r)$ the last equation is transformed to the equation (9) for $SU(2)$ monopole solution.

Existence of stationary solutions with a magnetic charge and a finite energy density is unexpected in a pure Yang-Mills theory since there is no such an analogue in linear field models including the Maxwell type gauge field. Remind, that a known 't Hooft-Polyakov monopole solution includes external Higgs scalar fields which regularize the monopole singularity at the origin. Besides, one can verify that the standard electroweak theory does not admit such regular stationary monopole solutions. So that, the quantum chromodynamics is a unique theory among the currently known theoretical field models of fundamental interactions which possesses such a surprising feature.

IV. STABILITY ANALYSIS

Stability of static localized solitonic solutions can be verified by checking whether the second variation of the energy with respect to small deformations becomes negative. If solution possesses a specific space symmetry one should take into account perturbations of most general field configuration as well. We will show that the stationary spherically symmetric monopole solution considered in the previous section is unstable against small axially-symmetric perturbations $Q_\mu^a(r, \theta, t)$ around the classical solution determined by the ansatz (10). In the case of stationary solutions with infinite total energy one can apply a similar approach to stability problem as in the case of solitons. We study an eigenvalue spectrum of the operator $\hat{K}_{\mu\nu}^{ab}$ defined by means of second variational derivatives of the classical action $S[A]$

$$\hat{K}_{\mu\nu}^{ab} Q_\nu^a = \lambda Q_\mu^a, \quad (19)$$

$$\hat{K}_{\mu\nu}^{ab} \equiv \frac{\delta^2}{\delta Q_\mu^a \delta Q_\nu^b} S[A_{cl} + Q]. \quad (20)$$

It is instructive to show that despite on infinite total energy and asymptotic oscillating behavior of the Lagrangian corresponding to the stationary monopole solution the classical action is finite and well-defined.

One can check that non-linear plane wave and stationary monopole solutions realize an absolute maximum of the classical action. Let us verify this for the stationary monopole solution. A solution for the stationary monopole can be represented in terms of Fourier series as follows

$$\psi(t, r) = C_0(r) + \sum_{n=1,2,3,\dots} P_n(r) \cos(nr) \sin(nt), \quad (21)$$

where one has only cosine functions in the decomposition due to the structure of the local solution, (13), and the radial functions $C_0(r), P_n(r)$ satisfy the boundary condition at the origin, $C_0(0) = 1, P_n(0) = 0$, and asymptotic solution (14), i.e., $P_n(r = \infty) = A_0$.

Let us consider for simplicity a leading mode approximation keeping only the first term in the series decomposition (21). The functions $C_0(r), P_1(r)$ can be decomposed in series in degrees of orthogonal polynomials. For our purpose to demonstrate that the variational method can be applied successfully to the classical action we choose a simple variational trial function for the monopole solution

$$\begin{aligned} \psi(t, r) = & 1 - \frac{(1 - a_0)r^2}{1 + b_0r^2} \\ & + A_0(1 - e^{-d_0r^2}) \cos(Mr) \sin(Mt), \end{aligned} \quad (22)$$

where the conformal scale factor M will be set to one in numeric calculations, a_0, b_0, A_0, d_0 are fitting number parameters. Note that in the case of the stationary monopole solution the convergence of the series (21) is very fast and the expression (22) provides a good interpolation function to the exact numeric solution. Substituting the trial function (22) into the action one can perform integration over the time in the interval $(0 \leq t \leq 2\pi)$, integration over the spherical angles (θ, φ) is trivial and leads to a number factor 4π . The obtained effective action A^{eff} is defined in one-dimensional space $(0 \leq r \leq \infty)$ and contains a divergent term A^{div} which originates from the kinetic terms in the original Lagrangian

$$A^{div} = 4\pi^2 \int dr A_0 \cos(2r). \quad (23)$$

A simple regularization by introducing an exponential factor $e^{-\varepsilon r}$ with an infinitesimally small number ε leads to a vanished value of the divergent term. The remaining part of the effective action is well-defined and integration over the radial coordinate can be easily performed numerically. With this, the action represents a regular function of the trial parameters (a_0, A_0, b_0, d_0) . The first two parameters represent a mean value and amplitude of the monopole solution in the asymptotic region, the parameters b_0, d_0 describe matching profile of the local and asymptotic solutions along the radial direction. One can fix one of two parameters a_0, A_0 since only one of them is independent, (see Fig. 3), all other parameters are found by variational procedure which finds an extremum the

action. The numeric results for a given asymptotic amplitude $A_0 = 0.56$ show that the classical action has an absolute maximum, $S^{max} = -0.168...$, with the following trial parameters values

$$\begin{aligned} a_0 &= 0.9982..., \\ b_0 &= 0.02588..., \\ d_0 &= 0.1786... \end{aligned} \quad (24)$$

The results are in a good qualitative agreement with the numeric results for the stationary monopole solution presented in Figs. 2,3.

Now we consider the stability of the monopole solution under axially-symmetric perturbations. We consider a small perturbation $Q(r, \theta, t)$ around the Abelian gauge potential A_φ^3 which describes a static Wu-Yang monopole within the framework of the ansatz (10)

$$A_\varphi^3 = \cos \theta + Q(r, \theta) \cos(Mt). \quad (25)$$

We constrain our study by consideration of perturbations with time dependent factor $\cos(Mt)$ with the same frequency M as one in the monopole solution. It is suitable to pass to standard notations in spherical coordinates defining the perturbation field $\tilde{Q}(r, \theta)$ of mass dimension

$$\tilde{Q}(r, \theta) = \frac{1}{r \sin \theta} Q(r, \theta). \quad (26)$$

For brevity of notations we denote the interpolation function $\psi(t, r)$, (22), as follows

$$\psi(t, r) = 1 + P_0(r) \cos(Mt). \quad (27)$$

In the case of small perturbations it is enough to keep only terms quadratic in $Q(r, \theta)$ in the classical action. Substituting the perturbed potential A_φ^3 , (25), into the classical action and performing integration over (t, φ) one obtains the following quadratic form for the operator $\hat{K}_{\mu\nu}^{ab}$, (20),

$$\begin{aligned} v\mathcal{L}^{(2)} &= 2\pi^2 \left[-r^2 \sin^2 \theta (\partial_r \tilde{Q})^2 - \sin \theta (\partial_\theta \tilde{Q})^2 \right. \\ &\quad - 2\tilde{Q}(\cos \theta \partial_\theta \tilde{Q} + r \sin \theta \partial_r \tilde{Q}) - \left(\csc \theta + \right. \\ &\quad \left. \left. (1 - M^2 r^2 + \frac{3}{4} P_0^2) \sin \theta \tilde{Q}^2 \right] \right], \end{aligned} \quad (28)$$

where $v \equiv r^2 \sin \theta$ is an integration volume in spherical coordinates. Varying the last expression with respect to \tilde{Q} one can write down explicitly the eigenvalue equation (19) for possible unstable modes

$$\begin{aligned} -r^2 \sin \theta \left[-\partial_r \partial_r - \frac{2}{r} \partial_r - \frac{1}{r} (\partial_\theta \partial_\theta + \cot \theta) + V \right] \tilde{Q} \\ = \lambda \tilde{Q}, \end{aligned} \quad (29)$$

$$V \equiv -M^2 + \frac{1}{r^2} \left(1 + \frac{1}{\sin^2 \theta} + \frac{3}{4} P_0(r) \right). \quad (30)$$

The equation resembles a Schrodinger type equation (up to opposite sign on the left hand side) with a quantum

mechanical potential V . Simple consideration shows that due to presence of the factor $-M^2$ the potential V is negative at large distance. This may cause appearance of negative eigenvalues for such a ‘‘Schrodinger’’ equation, or, equivalently, positive values for λ in the original eigenvalue equation (29). Corresponding eigenfunctions represent unstable perturbation modes which will increase the value of the non-perturbed action. Therefore, the spherical monopole solution will be unstable under axially-symmetric field perturbations. Note that in the case of spherically symmetric perturbations around the ansatz (10) a similar term $-M^2$ in the potential V does not imply negative eigenvalues. A detailed analysis shows that due to scaling invariance there are only zero perturbation modes corresponding to zero eigenvalues.

To solve numerically the equation (29) we impose the following boundary conditions for the perturbation field \tilde{Q} . A numeric solution corresponding to a positive eigenvalue closest to zero is presented in Fig. 4a. The unstable mode $\tilde{Q}_1(r, \theta)$ has oscillating behavior with an amplitude decreasing in radial direction as $\frac{1}{r^2}$.

$$\tilde{Q}(\infty, \theta) = 0, \quad \tilde{Q}(r, 0) = \tilde{Q}(r, \pi). \quad (31)$$

Note that since the eigenvalue equation (29) is linear one can normalize the function $\tilde{Q}_1(r, \theta)$ to any small number. We keep non-renormalized solutions $\tilde{Q}(r, \theta)$ in our numeric results. Numeric estimate of the second variation of the classical action confirms that it is positive. So, the perturbations increase the extremum value of the action showing instability of the spherically symmetric monopole.

Finite energy perturbation modes can be classified into two type field configurations with respect to reflection symmetry $\theta \rightarrow \pi - \theta$. The first mode \tilde{Q}_1 keeps its shape under the reflection and has even parity. An unstable mode corresponding to the next positive eigenvalue is depicted in Fig. 5a, and it represents odd parity field configuration. Unstable modes with larger positive eigenvalues correspond to presence of higher Fourier modes in polar angle (θ) . All regular unstable modes with appropriate asymptotic behavior lead to a finite total energy and finite second variation of the classical action. In particular, the energy density \mathcal{E}_1 corresponding to the mode \tilde{Q}_1 has integrable singularities at two points on Z -axis, $(r \simeq \pm 0.72, \theta = 0, \pi)$, and the energy density \mathcal{E}_2 corresponding to the mode \tilde{Q}_2 has a finite maximum along two rings with centers located on the Z -axis at finite distance from the origin, Fig. 5b.

Remind that $\tilde{Q}(r, \theta)$ is a perturbation around the Abelian gauge field $A_\varphi^3(r, \theta) = \cos \theta$ which describes a static Wu-Yang monopole. The presence of two types of unstable modes with energy density maximums located near the Z -axis at finite distance from the origin indicates to possible existence of two types of axially symmetric solutions which might correspond to monopole-monopole and monopole-antimonopole pairs.

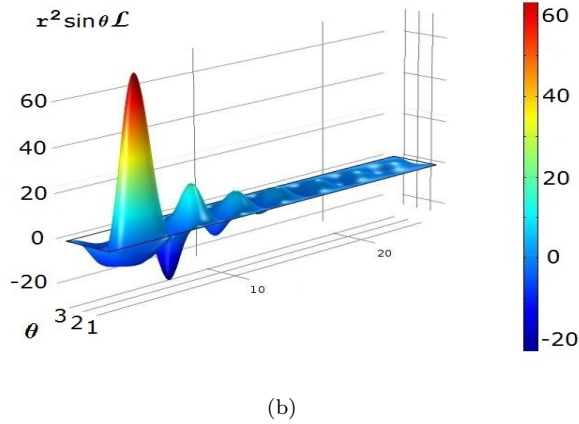
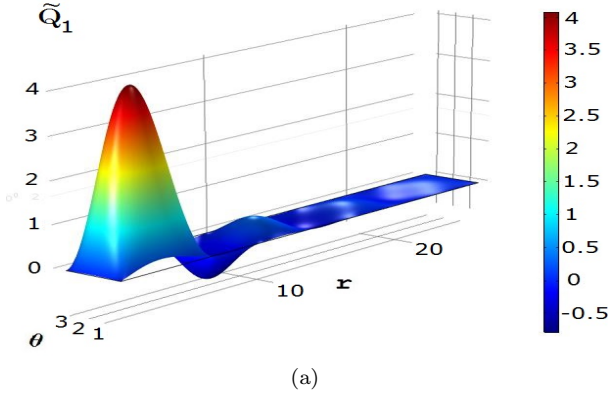


FIG. 4: (a) Perturbation mode $\tilde{Q}_1(r, \theta)$ of even parity corresponding to the eigenvalue $\lambda_1 = +1.498$; (b) an integral density $r^2 \sin \theta \mathcal{L}$ for the second variation of the classical action.

In general, in non-linear theories the stability of stationary solutions represents a non-trivial problem. We perform analysis of the stability in linear approximation for axially-symmetric perturbation around the Abelian gauge potential A_φ^3 . In this stage we may conclude that spherically symmetric monopole is unstable under small axially-symmetric perturbations.

V. GLUON SPIN DENSITY OPERATOR

The classical non-linear solutions considered in the present paper have common features: all of them have a mass scale parameter, longitudinal projections of color fields and zero spin density. It is known that gluons in QCD gain a dynamical mass due to non-perturbative quantum corrections. Appearance of the rest mass in classical spinless non-linear wave solutions implies possibility that corresponding spinless states may exist in the quantum theory. The propagating non-linear waves described in Section II can be treated as a longitudi-

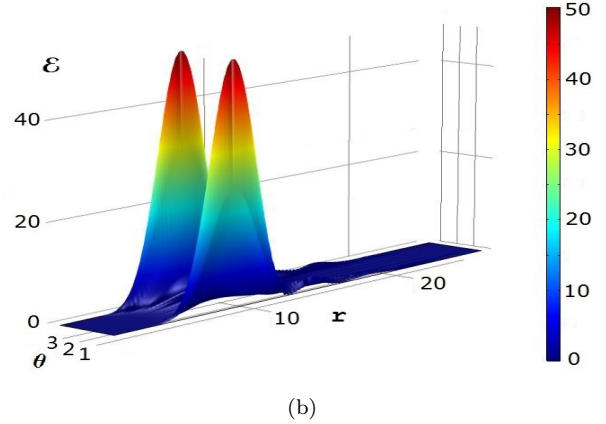
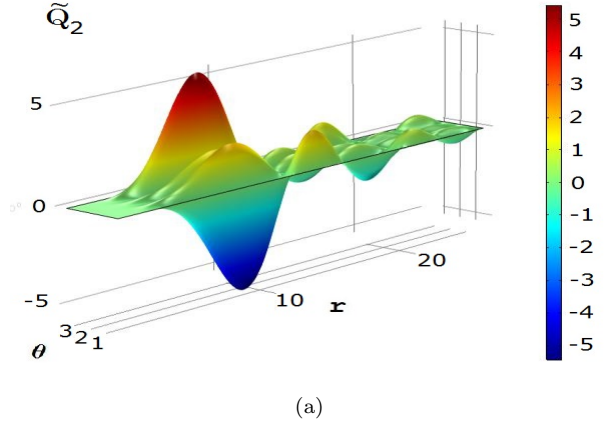


FIG. 5: (a) Perturbation mode $\tilde{Q}_2(r, \theta)$ of odd parity corresponding to the eigenvalue $\lambda_2 = +2.320$; (b) a respective energy density \mathcal{E} .

nal massive modes in addition to the Coleman transverse non-linear waves. It is clear, that such of a set of the non-linear transverse and longitudinal plane waves provides an attractive possibility for finding a proper non-perturbative quantization of QCD. This raises also a question of finding a consistent definition of the gluon spin density operator corresponding to the propagating spinless waves. A strict concept of the gluon spin operator includes several aspects, among them, the gauge invariance and Lorentz frame independence are the most important issues. Below we construct a unique gauge invariant and Lorentz frame independent definition of the gluon spin density operator for non-linear spinless waves at classical level.

Let us recall the main problems related to gauge invariant definitions of gluon spin and angular momentum operators in QCD. The standard canonical decomposition of the nucleon total angular momentum includes gauge non-invariant terms corresponding to quark and gluon

spin and orbital momentum operators [33]

$$J_{\mu\nu}^{\text{can}} = \int d^3x \left\{ \bar{\psi} \gamma^0 \frac{\Sigma_{\mu\nu}}{2} \psi - i \bar{\psi} \gamma^0 x_{[\mu} \partial_{\nu]} \psi - A_{a[\mu} F_{\nu]}^a \right. \\ \left. - F_{0\rho}^a x_{[\mu} \partial_{\nu]} A_a^\rho \right\}. \quad (32)$$

The expressions for the canonical spin density $S_{\mu\nu} = -A_{a[\mu} F_{\nu]}^a$ and angular momentum operators for quark and gluon are gauge non-invariant. It was assumed that a gauge invariant definition of the gluon spin operator in non-Abelian theory did not exist [33, 34]. A novel approach towards constructing a gauge invariant nucleon spin decomposition has been proposed in [35, 36]. The main idea in this approach is to separate physical degrees of freedom of gluon from pure gauge degrees of freedom, i.e., one splits the gauge potential into two parts

$$A_\mu^a = \mathcal{A}_\mu^a + \hat{A}_\mu^a, \quad (33)$$

where \hat{A}_μ^a is a physical gauge potential containing only physical degrees of freedom, and \mathcal{A}_μ^a is a pure gauge potential satisfying the pure gauge condition

$$\mathcal{F}_{\mu\nu}^a(\mathcal{A}) = 0. \quad (34)$$

Both potentials are constructed in terms of the original unconstrained vector potential A_μ^a . A key point is that the physical vector potential \hat{A}_μ^a transforms under the gauge transformation as a vector in adjoint representation of $SU(2)$ whereas the pure gauge potential \mathcal{A}_μ^a transforms as a gauge connection. This allows to re-write the canonical decomposition (32) in an explicit gauge invariant manner [35, 36]

$$J_{\mu\nu}^{\text{can}} = \int d^3x \left\{ \bar{\psi} \gamma^0 \frac{\Sigma_{\mu\nu}}{2} \psi - i \bar{\psi} \gamma^0 x_{[\mu} \mathcal{D}_{\nu]} \psi - \hat{A}_{a[\mu} F_{\nu]}^a \right. \\ \left. - F_{0\rho}^a x_{[\mu} (\mathcal{D}_{\nu]} \hat{A}_a^\rho - \mathcal{F}_{\nu]}{}^\rho{}_a(\mathcal{A})) \right\}, \quad (35)$$

where $\mathcal{D}_\mu \equiv \partial_\mu + A_\mu$ is a covariant derivative containing the pure gauge potential. Each term in (35), including the gluon spin density operator $\hat{S}_{\mu\nu} = -\hat{A}_{a[\mu} F_{\nu]}^a$, is explicitly gauge invariant due to covariant transformation properties of the physical gauge potential \hat{A}_μ^a . It is worth to stress that an equation defining the physical potential can be chosen by several ways, and the most important issue in the definition of the physical potential is a problem of uniqueness of gauge invariant and Lorentz frame independent definition of spin and momentum operators for quarks and gluon. Within the framework of the formalism proposed by Chen et al [35, 36] a Lorentz frame independent definition is proposed in [37], however, an explicit construction of such a definition is unknown since a perturbative solution of the equation for the physical gauge potential does not exist on the space of classical free plane wave solutions for the gluon field (see also the review [5] for current status of the problem).

Note that for massless particles a consistent concept of spin is given by the notion of helicity. In the case of

the Maxwell electrodynamics the gauge potential must satisfy two helicity gauge conditions,

$$A_0 = 0, \quad A_3 = 0, \quad (36)$$

to represent helicity eigenstates of the operator J_3 of a little group $E(2)$ which provides naturally the Lorentz invariance of the helicity operator [38]. The helicity conditions can be expressed in equivalent forms using various combinations of a generalized axial gauge condition

$$n^\mu A_\mu = 0, \quad (37)$$

where the constant vector n^μ specifies the temporal, axial or light-cone gauge conditions. The meaning of the helicity conditions is simple, they fix pure gauge degrees of freedom while keeping two transverse dynamic degrees of freedom of the gluon. Generalization of such description of the helicity operator to the case of non-Abelian theory has been done in [35, 36, 39, 40]. Since the helicity operator is Lorentz frame independent, all definitions satisfying the helicity conditions are consistent even though the defining equations for the physical gauge potential are not manifestly Lorentz invariant [5, 40].

In the case of non-linear massive plane waves described by the ansatz (2) the helicity conditions $A_0^a = 0$, $\partial^i A_i^a = 0$ are fulfilled only in the rest frame, $\beta = 0$. So that the definition of spin operator for such solutions based on the helicity conditions becomes Lorentz frame dependent. Solutions defined by the ansatz (8,10,16) satisfy the helicity conditions as well, $A_0^a = 0$, $A_r^a = 0$, but break the Lorenz gauge condition. In addition, one can verify that the canonical spin density for all these solutions vanishes identically. This implies that for massive spinless non-linear waves one should find a proper definition for gluon spin operator.

Note that the helicity conditions are consistent with the Gauss law, which guarantees consistency with all equations of motion. This is somewhat unexpected because the presence of helicity conditions means that one has two transverse polarizations of gluon which correspond to spin one particles. However, the non-linear wave type solutions considered above admit projections of color fields along the wave vector. This allows to interpret such non-linear waves as longitudinal modes, raising a question about the number of dynamic degrees of freedom for gluons. It is clear, that the origin of such a subtlety comes from the non-linearity of the gauge field strength expressed in terms of the gauge potential. One possible way to revise the notion of the gluon spin is to develop the old approach based on constructing gauge invariant quantities from the field strength [41, 42]. Note that, so far the known gauge invariant definitions of the gluon spin operator are conditioned by assumption that gluon has only two transverse dynamical degrees of freedom per each color degree of freedom. Such an assumption is based on the standard perturbative quantization of QCD. The presence of propagating massive wave solutions, which are essentially non-linear, implies that def-

inition of the gluon spin should be revised to include description of zero spin states.

Let us consider the definition for the physical gauge potential based on the Lorenz gauge type equation

$$(\mathcal{D}^\mu \hat{A}_\mu)^a = 0. \quad (38)$$

One should stress that the last equation looks similar to Lorenz gauge fixing condition in presence of a background field, however, it does not represent any gauge fixing in a fact. An explicit expression for the physical gauge potential \hat{A}_μ^a is provided as a solution to equation (38) in terms of a general initial gauge potential A_μ^a in such a manner that \hat{A}_μ^a still possesses full gauge freedom (it is transformed as $SU(2)$ vector in adjoint representation, not as a gauge connection). In practical use it is suitable to choose a real Lorenz gauge fixing condition for the physical potential with a trivial pure gauge counterpart, $\mathcal{A}_\mu^a = 0$. This is a key idea of the approach towards resolving the problem of gauge invariant definitions for gluon spin and orbital momentum operators [35, 36].

The equation (38) is unique (except the case of a generalized covariant Lorenz gauge type equation for the physical gauge potential which will be discussed below) among all possible conditions containing first derivatives and satisfying invariance under the Poincare group transformations. The Lorenz gauge type condition (38) was proposed in [37] as a transversality condition. In the case of Maxwell theory the definition of a gauge invariant photon spin operator based on the solution of the Lorenz type equation (38) encounters a well-known problem of incompleteness of the Lorenz gauge [40]. Namely, in the Lorenz gauge one has still a residual symmetry which implies that equation of motion for the temporal component of the gauge potential admits unphysical propagating modes. To fix such a symmetry one has to impose an additional gauge condition. We show that in the non-Abelian $SU(2)$ gauge theory the definition of a gauge invariant spin operator based on the equation (38) is possible due to absence of the residual symmetry for the class of non-linear wave solutions described by ansatz (2,8,10,16). To verify this we construct a formal series operator expansion for such a solution using the perturbative method. The solution for the physical potential \hat{A}_μ can be found as a series

$$\hat{A}_\mu^a = \hat{A}_\mu^{a(0)} + g\hat{A}_\mu^{a(1)} + g^2\hat{A}_\mu^{a(2)} + \dots \quad (39)$$

Expressing the pure gauge potential in terms of the physical one, $\mathcal{A}_\mu^a = A_\mu^a - \hat{A}_\mu^a$, one can find a solution to the equations (34) and (38) in the leading order approxima-

tion

$$\begin{aligned} \hat{A}_\mu^a &= P_{\perp\mu}{}^\nu A_\nu^a \\ &+ g \frac{1}{\square} P_{\perp\mu}{}^\nu \epsilon^{abc} \left(\partial^\rho (A_\rho^b A_\nu^c) + \hat{A}_\rho^{b(0)} \partial^\rho \hat{A}_\nu^{c(0)} \right. \\ &\quad \left. - \partial^\rho (A_\rho^b \hat{A}_\nu^{c(0)}) + \partial^\rho A_\nu^b \hat{A}_\rho^{c(0)} \right) \\ &+ \hat{A}_\mu^{a \text{ long}}, \\ \hat{A}_\mu^{a \text{ long}} &= -g \epsilon^{abc} \left(\frac{1}{\square} \partial^\rho A_\rho^b \right) (P_{\perp\mu}{}^\sigma A_\sigma^c), \\ \hat{A}_\mu^{(0)} &= P_{\perp\mu}{}^\nu A_\nu^a, \end{aligned} \quad (40)$$

where we use the transverse projectional operator $P_{\perp\mu}{}^\nu = \delta_\mu^\nu - \frac{\partial_\mu \partial^\nu}{\square}$. Note that the solution (40) for the physical gauge potential includes an expression $\hat{A}_\mu^{a \text{ long}}$ containing the longitudinal part, $\partial^\rho A_\rho^a$, of the unconstrained gauge potential. The fact that a gauge invariant expression (40) is given in terms of the gauge potential, not in terms of the field strength, reflects the property of the non-Abelian gauge theory where the field strength does not determine all gauge invariant quantities as it occurs in the Abelian gauge theory. One should stress that non-locality appearing in the solution for the physical gauge potential is unphysical. Such a non-locality disappears when we impose the real Lorenz gauge fixing condition on the physical potential; in this case the pure gauge potential vanishes identically and the physical gauge potential coincides with the initial general gauge potential A_μ^a . So that the non-local expression for the gluon spin density operator reduces to the standard local expression for the canonical spin density with A_μ^a satisfying the usual Lorenz gauge condition.

Let us verify that the Lorenz gauge condition is complete and does not possess a residual symmetry on a space of the non-linear wave type solutions. A small gauge variation of the Lorenz gauge condition leads to the following equation for the gauge parameter λ^a

$$(\partial^\mu D_\mu \lambda)^a = 0. \quad (41)$$

For simplicity we consider a class of plane wave solutions provided by the ansatz (2). One can verify that any solution to equation (41) for the parameters λ^a does not belong to the same class of the non-linear plane wave solutions determined by the ansatz (2).

Let us consider a counter example, a generalized Lorenz gauge $(D^\mu A_\mu)^a = 0$ which contains a full covariant derivative D^μ including the gauge potential. Under small gauge variation it leads to a covariant D'Alembert equation

$$(D^\mu D_\mu \lambda)^a = 0. \quad (42)$$

Applying the ansatz (2) for homogeneous solutions, one obtains the following system of equations

$$\begin{aligned} \ddot{\lambda}_1 + g\lambda_1(\phi_2^2 + \phi_3^2) &= 0, \\ \ddot{\lambda}_2 + g\lambda_2(\phi_3^2 + \phi_1^2) &= 0, \\ \ddot{\lambda}_3 + g\lambda_3(\phi_3^2 + \phi_1^2) &= 0. \end{aligned} \quad (43)$$

It is clear, that the above system of equations on the space of the non-linear plane wave solutions (7) admits solutions for λ^a exactly of the same type. This implies appearance of a residual symmetry at the level of a full nonlinear theory and makes the covariant Lorenz gauge incomplete. So that, a defining equation for the physical gauge potential based on use of a generalized covariant Lorenz gauge type equation can not provide a self-consistent definition of a gauge invariant and Lorenz covariant gluon spin operator.

Summarizing, we propose a new definition for the gluon spin operator for a class of non-linear plane wave solutions which admit a non-vanishing mass and a total spin zero. The definition for the physical gauge potential is based on the equation (38) and we have provided an explicit construction for the physical gauge potential, (40), which leads to unique gauge invariant and Lorenz frame independent definition of the gluon spin operator for massive spinless non-linear plane waves. Note, that equation (38) was suggested in [37] as a possible definition of the gluon spin operator for massless gluons with helicity ± 1 . However, the equation (38) does not admit solutions due to presence of the residual symmetry. So that, a consistent definition of the gluon spin operator for massless gluons is possible only on the basis of the helicity equations (36, 37).

VI. CONCLUSION

We propose a new stationary generalized monopole solution which can be treated (at least in the singular Abelian gauge) as a system of the static Wu-Yang monopole and off-diagonal gluon. An essential feature of that solution is that it possesses a finite energy density everywhere in the whole space.

For a class of propagating non-linear waves described by the ansatz (2) we construct a unique gauge invariant and Lorenz frame independent definition for the gluon spin operator based on a Lorenz gauge type equation for the physical gauge potential. Note that in the case of the stationary monopole solution defined by the ansatz (10) the canonical spin density vanishes identically as well. However, the Lorenz gauge condition is not fulfilled

$$\partial^\mu A_\mu^a = \delta^{a,2} \frac{1}{r^2} \cot \theta \psi(r, t). \quad (44)$$

This is not surprising, since the stationary monopole solution does not represent a stationary propagating solution like a plane wave. Passing to arbitrary Lorenz frame such a solution will represent a moving lump with a maximal energy density around its center mass. Due to this, the definition of the gluon spin operator for the stationary generalized Wu-Yang monopole solutions and for non-stationary solutions corresponding to the ansatz (10) is based on a generalized axial gauge condition as for the massless gluon [40]. One should note that, since the notion of spin has a quantum mechanical origin, one has to perform properly the quantization of the gluon field and construct the algebra of quantized angular momentum and spin operators. Since for the case of non-linear solutions the angular momentum and spin operators are non-linear operator functions of the quantized gauge potential A_μ^a , one expects that such a quantum algebra can be realized as a deformation or a non-linear representation of the standard Lie algebra of $SO(3)$.

An important issue is to study possible physical implications of non-linear wave type solutions in QCD. It has been shown that non-linear type I plane waves, (7), provide a simple estimate of glueball spectrum in qualitative agreement with lattice calculation [14]. Our stability analysis shows that the stationary spherically symmetric monopole is unstable under axially-symmetric perturbations. This indicates to existence of stationary monopole and monopole-antimonopole pair solutions. This issue will be considered in a subsequent paper [30].

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